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Super local edge anti-magic total coloring of paths and its derivation

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Abstract

Suppose $G(V, E)$ be a graph and suppose u, v, x be vertices of graph G. A bijection $f : V \cup E \to \{1, 2, 3, ..., |V(G)| + |E(G)|\}$ is called super local edge antimagic total coloring if for any adjacent edges uv and vx in $E(G)$, $w(uv) \neq w(vx)$, which $w(uv) = f(u) + f(uv) + f(v)$ for $f(V) = \{1, 2, 3, ..., |V(G)|\}$. By giving G a labeling f, we denotes the minimum number of distinct weight of edges needed in G as $\gamma_{sleat}(G)$. In this study, we proved the γ_{sleat} of paths and its derivation.

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1. Introduction

For a graph G, a bijection $f : V \cup E \to \{1, 2, 3, ..., |V(G)| + |E(G)|\}$ assigns each edges to a particular distinct number from 1 up to $|V(G)| + |E(G)|$. Suppose u, v, x be vertices of graph G. Weight of an edge $w(uv)$ is defined as $w(uv) = f(u) + f(uv) + f(v)$ for every edge uv in G. A bijection f is called local edge antimagic total coloring if any adjacent edges uv and vx, $w(uv) \neq w(vx)$. For every distinct weight, we denote them as distinct colors. The local edge antimagic total chromatic number of G, $\gamma_{leat}(G)$, is the minimum number of colors for edges taken over all colorings induced by local antimagic total labelings of G. If vertices of G are assigned smaller labels, then we call f as super local edge antimagic total coloring of G, and the minimum number of colors as γ_{sleat}. For convenience, we will use the abbreviation SLEAT as super local

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edge anti-magic total. A SLEAT colorable graph is a graph that admits SLEAT coloring. Let $\gamma(G)$ be the edge chromatic number of G. We can observe that $\Delta(G) \leq \gamma(G)$, where $\Delta(G)$ is a maximum degree in graph G. A compilation of observation concludes that $\Delta(G) \leq \gamma(G) \leq \gamma_{leat}(G) \leq \gamma_{sleat}(G)$ for an arbitrary graph G.

Ringel firstly introduced the term of antimagic labelings [6]. There are many researchs conducted in antimagic labelings and its variation based on the survey of Gallian [5]. One of many variations is local antimagic labelings.
Arumugam et al. introduced local antimagic as vertex local antimagic edge labelings [3]. Followed by labels addition for the vertex, vertex local antimagic total labelings, one described in by Kurniawati et al. [8]. The analog rises, edge local antimagic total labelings, explained by Agustin et al. in [1] that includes path graph. If every vertex labels are smaller than edge labels, then it is a super local edge antimagic total labelings, which Agustin et al. [2] and Kurniawati et al. [7] describe in some other graph.

In this paper, we study super local edge anti-magic total coloring of path graph, path with edge(s) addition which forms unicycle, connecting two disjoint path, and amalgamation of star. If $\gamma(G) = \gamma_{sleat}(G)$, then by the prior inequality, $\gamma_{leat}(G) = \gamma_{sleat}(G)$. In other words, studies about local edge anti-magic total (LEAT) coloring in all graphs that satisfy $\gamma(G) = \gamma_{sleat}(G)$ are not necessary.

2. Main Results

2.1. Path Graphs

First, we are going to establish SLEAT for paths. This theorem is used in every proceeding theorems.

Theorem 2.1. *Let $n \geq 2$ be integer and P_n be a path with n vertices. $\gamma_{sleat}(P_n) = 2$.*

Proof. Let $V(P_n) = \{v_i | 1 \leq i \leq n\}$ and $E(P_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\}$. Since $\Delta(P_n) = 2$, $\gamma_{sleat}(P_n) \geq 2$. To show $\gamma_{sleat}(P_n) \leq 2$, define $f : V(P_n) \cup E(P_n) \rightarrow \{1, 2, ..., |V(P_n)|+|E(P_n)|\}$.

Case 1. n is odd.

Labels vertices and edges as follows

$$f(v_i) = \begin{cases} \frac{i}{2}, & \text{if} \quad i \text{ is even,} \\ \frac{n+i}{2}, & \text{if} \quad i \text{ is odd,} \end{cases}$$

$$f(v_i v_{i+1}) = \begin{cases} 2n - i + 1, & \text{if} \quad i \text{ is even,} \\ 2n - i - 1, & \text{if} \quad i \text{ is odd.} \end{cases}$$

Hence, we get

$$w(v_i v_{i+1}) = f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) = \begin{cases} \frac{5n+1}{2} + 1, & \text{if} \quad i \text{ is even,} \\ \frac{5n+1}{2} - 1, & \text{if} \quad i \text{ is odd.} \end{cases}$$

Case 2. n is even.

Labels vertices and edges as follows

$$f(v_i) = \begin{cases} n, & \text{if} \quad i = 1, \\ \frac{n+i}{2} - 1, & \text{if} \quad i \text{ is even}, \\ \frac{i-1}{2}, & \text{if} \quad \text{odd } i > 1, \end{cases}$$

$$f(v_i v_{i+1}) = \begin{cases} n+1, & \text{if} \quad i = 1, \\ 2n - i, & \text{if} \quad i \text{ is even}, \\ 2n - i + 2, & \text{if} \quad \text{odd } i > 1. \end{cases}$$

Hence, we get

$$w(v_i v_{i+1}) = f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) = \begin{cases} \frac{5n}{2} - 1, & \text{if} \quad i \text{ is even}, \\ \frac{5n}{2} + 1, & \text{if} \quad i \text{ is odd}. \end{cases}$$

Since in every case there is only two distinct w, therefore f is a SLEAT labeling of P_n with $\gamma_{sleat}(P_n) = 2$. □

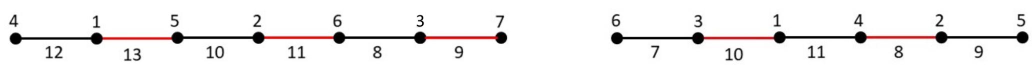

Figure 1: SLEAT coloring of P_7 and P_6

2.2. Unicycle graph

2.2.1. Path with an edge addition

Now, we start to add variation by adding an edge for path, forming a unicycle graph. We want to exclude the possibility of forming a regular cycle.

Theorem 2.2. *Let $n \geq 3$ be integer, P_n be path, and $G = P_n + v_i v_j$, where $v_i, v_j \in V(P_n)$, with i and j integers $1 \leq i < j \leq n$ and, except $(i, j) = (1, n)$. $\gamma_{sleat}(G) = 3$.*

Proof. Since $\Delta(G) = 3$, $\gamma_{sleat}(G) \geq 3$. To show that $\gamma_{sleat}(G) \leq 3$, suppose g is a labeling of the unicycle graph and f is a SLEAT coloring of P_n given in the proof of Theorem 2.1. We divide the proof into 4 subcases.

Case 1. n is odd.

Subcase 1.1. Either i or j is odd. Labels vertices and edges as follows

$$g(P_n) = f(P_n),$$

$$g(v_i v_j) = 2n.$$

Hence, we get

$$w(v_iv_j) = g(v_i) + g(v_iv_j) + g(v_j) = \begin{cases} 3n + \frac{i+j}{2}, & \text{if} \quad i \text{ and } j \text{ are odd,} \\ \frac{5}{2}n + \frac{i+j}{2}, & \text{if} \quad i \text{ or } j \text{ is odd.} \end{cases}$$

Therefore, we get a minimum value of $w(v_iv_j) \geq \frac{5n+i+j}{2}$. Now, we will show that $w(v_iv_j)$ has different value with $w(v_iv_{i+1})$. It is clear that $w(v_iv_{i+1})$ has a maximum value of $\frac{5n+1}{2} + 1$. By the premise, $i + j$ has the smallest possible value attainable $i + j \geq 4$. We observe that

$$w(v_iv_j) \geq \frac{5n+4}{2} > \frac{5n+1}{2} + 1,$$

which implies that $w(v_iv_j)$ is a third distinct weight.

Sub-case 1.2. i and j are even. Labels vertices and edges as follows

$$g(V(P_n)) = f(V(P_n)),$$

$$g(E(P_n)) = f(E(P_n)) + 1,$$
$$g(v_iv_j) = n + 1.$$

Since every labels of edges is increased by 1, we will have a new weight of

$$w(v_iv_j) = g(v_i) + g(v_iv_j) + g(v_j),$$

$$w(v_iv_{i+1}) = \begin{cases} \frac{5n+1}{2} + 2 & \text{for} \quad i \text{ is even,} \\ \frac{5n+1}{2} & \text{for} \quad i \text{ is odd.} \end{cases}$$

and $w(v_iv_j) = n + 1 + \frac{i+j}{2}$. Now, we will show that $w(v_iv_j)$ has different value with $w(v_iv_{i+1})$. It is clear that $w(v_iv_{i+1})$ has a minimum value of $\frac{5n+1}{2}$. By the premise, $i + j$ has the largest possible value attainable $i + j \leq 2n - 4$. We observe that

$$w(v_iv_j) \leq n + 1 + \frac{2n-4}{2} = 2n - 1 < \frac{5n+1}{2},$$

and $w(v_iv_j)$ is a third distinct weight.

Case 2. n is even.

Subcase 2.1. Either i or j is equal to 1 or even. Labels vertices and edges as follows

$$g(P_n) = f(P_n),$$

$$g(v_iv_j) = 2n.$$

Hence, we get possible weights

$$w(v_iv_j) = \begin{cases} 3n + \frac{i+j}{2} - 2, & \text{if} \quad i \text{ and } j \text{ are even}, \\ \frac{7}{2}n + \frac{j}{2} - 1, & \text{if} \quad i = 1 \text{ and } j \text{ is even}, \\ \frac{5}{2}n + \frac{i+j}{2} - \frac{3}{2}, & \text{if} \quad i \text{ is even and } j \text{ is odd}, \\ \frac{5}{2}n + \frac{i+j}{2} - \frac{3}{2}, & \text{if} \quad i \text{ is odd and } j \text{ is even}, \\ 3n + \frac{j}{2} - \frac{1}{2}, & \text{if} \quad i = 1 \text{ and } j \text{ is odd}. \end{cases}$$

We will show that $w(v_iv_j)$ has different value with $w(v_iv_{i+1})$. It is clear that $w(v_iv_{i+1})$ has a maximum value of $\frac{5n}{2} + 1$. A minimum value for these possible weights is needed to be determined.

If both i and j are even, then $i + j$ has smallest value attainable $i + j \geq 2 + 4 = 6$, such that $w(v_iv_j) \geq 3n + 1$. If $i = 1$ and j is even, then j has smallest value attainable $j \geq 4$, such that $w(v_iv_j) \geq \frac{7}{2}n + 1 = \frac{5}{2}n + n + 1$. If i is even and j is odd, then $i + j$ smallest value attainable $i + j \geq 2 + 5 = 7$, such that $w(v_iv_j) \geq \frac{5}{2}n + 2$. If $i = 1$ and j is odd, then j smallest value attainable $j \geq 3$, such that $w(v_iv_j) \geq 3n + 1$.

It can be seen that $w(v_iv_j)$ has a minimum value of $\frac{5}{2}n + 2 \geq \frac{5}{2}n + 1$, since n is even. It means $w(v_iv_j)$ is a third distinct weight.

Subcase 2.2. i and j are odd, and both of them does not equal to 1. Labels as follows

$$g(V(P_n)) = f(V(P_n)),$$
$$g(E(P_n)) = f(E(P_n)) + 1,$$
$$g(v_iv_j) = n + 1.$$

since every labels of edges is incremented by 1, we will have a new weight of

$$w(v_iv_j) = g(v_i) + g(v_iv_j) + g(v_j),$$

$$w(v_iv_{i+1}) = \begin{cases} \frac{5n}{2}, & \text{if} \quad i \text{ is even}, \\ \frac{5n}{2} + 2, & \text{if} \quad i \text{ is odd}. \end{cases}$$

and $w(v_iv_j) = n + \frac{i+j}{2}$. We will show that this $w(v_iv_j)$ has different value with $w(v_iv_{i+1})$. It is clear that $w(v_iv_{i+1})$ has a minimum value of $\frac{5n}{2}$. By the premise, $i + j$ has the largest attainable value $i + j \leq 2n - 4$. We observe that $w(v_iv_j) \leq n + \frac{2n-4}{2} = 2n - 2 < \frac{5n}{2}$. Therefore, $w(v_iv_j)$ is a third distinct weight.

Since in every case $\gamma_{sleat}(G) \leq 3$, we can deduce that g is SLEAT of G with $\gamma_{sleat}(G) = 3$. □

2.2.2. Path with some edges addition

This method of edge addition can be done multiple times if it meets certain premises. We need to introduce a convenient even function, defined as follows

$$\text{even}(n) = \begin{cases} 1, & \text{if} \quad n \text{ is even}, \\ 0, & \text{if} \quad n \text{ is odd}. \end{cases}$$

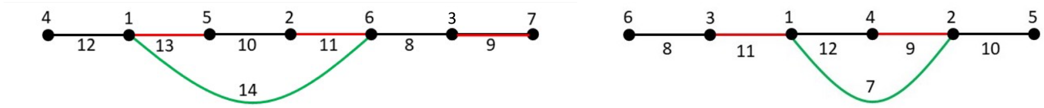

Figure 2: SLEAT coloring of $G = P_7 + v_2v_5$ and $G = P_6 + v_3v_5$

Theorem 2.3. *Let* $n \geq 4$ *be integer,* P_n *be path, and* $G = P_n + \{v_{i-k}v_{j-k} | 0 \leq k \leq p, 0 \leq p \leq \min\{i - 1 + \mathit{even}(n), j - i - 1\}\}$ *where* $v_{i-k}, v_{j-k} \in V(P_n)$*,* i *and* j *satisfy* $1 \leq i < j \leq n$*,* $i + j$ *is odd, and* $(i, j) \neq (1, n)$*.* $\gamma_{sleat}(G) = 3$

Proof. Since $\Delta(G) = 3$, $\gamma_{sleat}(G) \geq 3$. To show that $\gamma_{sleat}(G) \leq 3$, suppose g is a labeling of G and f is a SLEAT coloring of P_n given in the proof of Theorem 2.1.
Labels vertices and edges as follows

$$g(P_n) = f(P_n),$$

$$g(v_{i-k}v_{j-k}) = 2n + k.$$

The weight of edge are the following

$$w(v_{i-k}v_{j-k}) = g(v_{i-k}) + g(v_{i-k}v_{j-k}) + g(v_{j-k}) = \begin{cases} \frac{5}{2}n + \frac{i+j}{2}, & \text{if } n \text{ is odd, } i + j \text{ is odd,} \\ \frac{5}{2}n + \frac{i+j}{2} - \frac{3}{2}, & \text{if } n \text{ is even, } i + j \text{ is odd.} \end{cases}$$

We will show that $w(v_{i-k}v_{j-k})$ has different value with $w(v_iv_{i+1})$. It is clear that $w(v_iv_{i+1})$ has a maximum value of $\frac{5n+1}{2} + 1$. A minimum value for these possible weights is needed to be determined.

If both n and $i + j$ are odd, then smallest value attainable $i + j \geq 1 + 3 = 4$, such that $w(v_{i-k}v_{j-k}) \geq \frac{5}{2}n + 2$. If n is even, $i \neq 1$, and $i + j$ is odd, then smallest value attainable $i + j \geq 2 + 5 = 7$, such that $w(v_{i-k}v_{j-k}) \geq \frac{5}{2}n + 2$.

Since $n \geq 4$, $w(v_{i-k}v_{j-k})$ has a minimum value of $\frac{5}{2}n + 2 > \frac{5n+1}{2} + 1$. As a result, $w(v_{i-k}v_{j-k})$ is a third distinct weights. Therefore, g is a SLEAT of G with $\gamma_{sleat}(G) = 3$. □

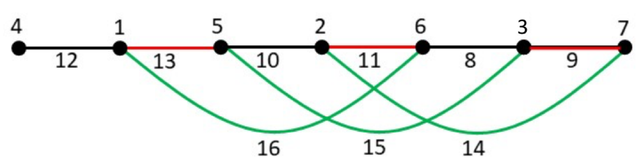

Figure 3: SLEAT coloring of $G = P_7 + \{v_{4-k}v_{7-k} | 0 \leq k \leq 2\}$

2.3. Connecting disjoint paths

Another kind of graph that is observed in this study is some disjoint paths with addition of edge(s) that connects some paths. Two kinds of this family of graph are hedge graph and hedgerow graph. These two graph will be defined in the proceeding subsections.

2.3.1. Hedge graph

To start, we observe two disjoint paths with addition of one edge that connects the paths. Such graph is called hedge graph Hd. Before giving the formal definition of hedge graph, we want to exclude the possibility of connection that forms a simple path. For arbitrary set A, B, we use the cartesian products of the two sets $A \times B = \{(a, b)|a \in A, b \in B\}$ in order to have simple notations. Let P_m and P_n be path with $V(P_m) = \{v_k|1 \leq k \leq m\}$ and $V(P_n) = \{u_l|1 \leq l \leq n\}$. We define $Hd(m, n, i, j)$ as follows

$$V(Hd(m, n, i, j)) = V(P_m) \cup V(P_n),$$

$$E(Hd(m, n, i, j)) = E(P_m) \cup E(P_n) \cup \{v_i u_j|1 \leq i \leq m, 1 \leq j \leq n, (i, j) \notin (1, n) \times (1, m)\}.$$

Theorem 2.4. *For integers $n \geq m \geq 3$, if $G = Hd(m, n, i, j)$, then $\gamma_{sleat}(G) = 3$.*

Proof. Since $\Delta(G) = 3$, $\gamma_{sleat}(G) \geq 3$. To show that $\gamma_{sleat}(G) \leq 3$, suppose g is a labeling of G and f is a SLEAT coloring of P_n given in the proof of Theorem 2.1.

Case 1. $i \neq j$.

Add an edge v_1u_1 to form a P_{m+n}, apply labeling f in a path P_{m+n} from v_n to u_m, then remove v_1u_1. Now, g is a labelings that follows

$$g(v_i u_j) = f(v_1 u_1),$$

$$g(V(G)) = f(V(P_{m+n})),$$

$$g(E(G)\backslash\{v_i u_j\}) = f(E(P_{m+n})\backslash\{v_1 u_1\}).$$

If $n + m$ is odd, we have the following weight

$$w(v_i v_{i+1}) = w(u_i u_{i+1}) = \begin{cases} \frac{5n+1}{2} + 1, & \text{if } \ i \text{ is even,} \\ \frac{5n+1}{2} - 1, & \text{if } \ i \text{ is odd.} \end{cases}$$

As for $n + m$ is even, we have the following weight

$$w(v_i v_{i+1}) = w(u_i u_{i+1}) = \begin{cases} \frac{5n}{2} - 1, & \text{if } \ i \text{ is even,} \\ \frac{5n}{2} + 1, & \text{if } \ i \text{ is odd.} \end{cases}$$

We get $g(v_1) + g(v_i u_j) + g(u_1) = w(v_i v_{i+1}) = w(u_i u_{i+1})$ for even i. Now, we can observe the partition $\{g(v_i)\}$ and $\{g(u_j)\}$ for arbitrary i and j, to find that there is no pair of $(g(v_i), g(u_j))$ such that $g(v_i) + g(u_j) = g(v_1) + g(u_1)$ which satisfies $i \neq j$. Hence, $w(v_i u_j) \neq w(v_i v_{i+1})$ and $w(v_i u_j) \neq w(v_i v_{i+1})$ for even i.

Next, we get $g(v_1) + g(v_iu_j) + g(u_1) - 2 = w(v_iv_{i+1}) = w(u_iu_{i+1})$ or $g(v_1) + g(v_iu_j) + g(u_1) + 2 = w(v_iv_{i+1}) = w(u_iu_{i+1})$ for odd i. Again, we can observe the partition $\{g(v_i)\}$ and $\{g(u_j)\}$ for arbitrary i and j, to find that there is no pair of $(g(v_i), g(u_j))$ such that $g(v_i)+g(u_j) = g(v_1) + g(u_1) - 2$ or $g(v_i) + g(u_j) = g(v_1) + g(u_1) + 2$. Hence, $w(v_iu_j) \neq w(v_iv_{i+1})$ and $w(v_iu_j) \neq w(v_iv_{i+1})$ for odd i.

Case 2. $i = j$, and $n \neq m$.

Add an edge v_nu_m to form a P_{m+n}, apply labeling f to path P_{m+n} from v_1 to u_1, then remove v_nu_m. Now, g is a labelings that follows

$$g(v_iu_j) = f(v_nu_m),$$
$$g(V(G)) = f(V(P_{m+n})),$$
$$g(E(G)\backslash\{v_iu_j\}) = f(E(P_{m+n})\backslash\{v_nu_m\}).$$

If $n + m$ is odd, we have the following weight

$$w(v_iv_{i+1}) = w(u_{i+1}u_{i+2}) = \begin{cases} \frac{5n+1}{2} + 1, & \text{if } \ i \text{ is even,} \\ \frac{5n+1}{2} - 1, & \text{if } \ i \text{ is odd.} \end{cases}$$

With the addition of $w(u_1u_2) = \frac{5n+1}{2} + 1$. As for $n + m$ is even, we have the following weight

$$w(v_iv_{i+1}) = w(u_iu_{i+1}) = \begin{cases} \frac{5n}{2} - 1, & \text{if } \ i \text{ is even,} \\ \frac{5n}{2} + 1, & \text{if } \ i \text{ is odd.} \end{cases}$$

We get $g(v_n) + g(v_iu_j) + g(u_m) = w(v_iv_{i+1}) = w(u_{i+1}u_{i+2}) = w(u_1u_2)$ for n and i that have the same parity. Now, we can observe the partition $\{g(v_i)\}$ and $\{g(u_i)\}$ for arbitrary i, to find that there is no pair of $(g(v_i), g(u_i))$ such that $g(v_i)+g(u_i) = g(v_n)+g(u_m)$. Hence, $w(v_iu_j)$ does not equal any of $w(v_iv_{i+1}), w(u_{i+1}u_{i+2})$, or $w(u_1u_2)$.

Next, we get $g(v_n) + g(v_iu_j) + g(u_m) - 2$ or $g(v_1) + g(v_iu_j) + g(u_1) + 2$ to be equal with $w(v_{i+1}v_{i+2}) = w(u_iu_{i+1}) = w(v_1v_2)$ for n and i have different parity. Again, we can observe the partition $\{g(v_i)\}$ and $\{g(u_i)\}$ for arbitrary i, to find that there is no pair of $(g(v_i), g(u_i))$ such that $g(v_i) + g(u_i) = g(v_n) + g(u_m) - 2$ or $g(v_i) + g(u_i) = g(v_n) + g(u_m) + 2$. Hence, $w(v_iu_j)$ does not equal any of $w(v_{i+1}v_{i+2}), w(u_iu_{i+1})$, or $w(v_1v_2)$.

Case 3. $i = j$, and $n = m$, except $n = 3$.

Since the graph with $i = 2$ is isomorphic with $i = n - 1$, it is only necessary to determine labels for $i \geq 3$.

Add an edge v_1u_1 to form a path P_{m+n}, apply labeling f to P_{m+n} from v_n to u_m, then remove v_1u_1.
If n is odd, g is a labelings that follows

$$g(v_iu_j) = f(u_1u_2),$$

$$g(u_1u_2) = f(v_1u_1),$$
$$g(V(G)) = f(V(P_{m+n})),$$
$$g(E(G)\backslash\{v_iu_j \cup u_1u_2\}) = f(E(P_{m+n})\backslash\{v_1u_1 \cup u_1u_2\}).$$

If n is even, g is a labelings that follows

$$g(v_iu_j) = f(v_1v_2),$$
$$g(v_1v_2) = f(v_1u_1),$$
$$g(V(G)) = f(V(P_{m+n})),$$
$$g(E(G)\backslash\{v_iu_j \cup v_1v_2\}) = f(E(P_{m+n})\backslash\{v_1u_1 \cup v_1v_2\}).$$

By preceding labelings, we have the following weight

$$w(v_{i+1}v_{i+2}) = w(u_{i+1}u_{i+2}) = \begin{cases} \frac{5n}{2} - 1, & \text{if } \ i \text{ is even,} \\ \frac{5n}{2} + 1, & \text{if } \ i \text{ is odd.} \end{cases}$$

In addition, if n is odd, $w(v_1v_2) = w(v_{i+1}v_{i+2})$ and $w(u_1u_2) = w(v_iu_j)$. While if n is even, $w(u_1u_2) = w(u_{i+1}u_{i+2})$ and $w(v_1v_2) = w(v_iu_j)$. By case 1, it is already proven that $w(v_iu_j)$ is distinct.

Case 4. $i = j = 2$, and $n = m = 3$.

This is the only one specific graph that does not belong to any preceding case. Thus, we can simply do an enumeration for completing the theorem. Labels the graph G as in Figure 4.

Since every possible graph is covered, therefore $\gamma_{sleat}(Hd(m, n, i, j)) = 3$. □

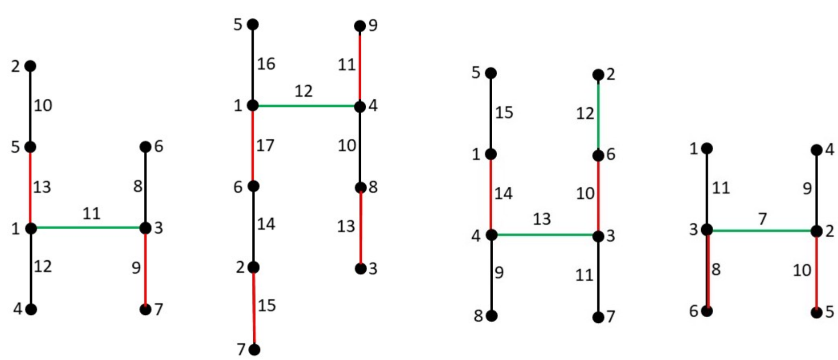

Figure 4: SLEAT coloring of $Hd(4, 3, 3, 2)$, $Hd(5, 4, 2, 2)$, $Hd(4, 4, 3, 3)$, and $Hd(3, 3, 2, 2)$

2.3.2. Hedgerow

Suppose we have m disjoint paths with length of n. Every neighbouring path is connected with an edge that is uniform through every two connected paths. Such graph is called hedgerow graph Hr. Let mP_n be the disjoint union of m number of P_n with $V(mP_n) = \{v_l^k | 1 \leq l \leq n, 1 \leq k \leq m\}$. Formally, we define hedgerow graph $Hr(m, n, i, j)$ as follows

$$V(Hr(m, n, i, j)) = V(mP_n),$$

$$E(Hr(m, n, i, j)) = E(mP_n) \cup \{v_i^k v_j^{k+1} | 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq m-1\}.$$

We start an attempt to find hedgerow for $m \geq 3$. We only find the value of γ_{sleat} for hedgerow graph for some (m, n, i, j).

Theorem 2.5. *Let integer $m \geq 2$ and integer $n \geq 2$. For $G \cong Hr(m, n, i, j)$,*

1. *If n is even, $i \neq j$, and $i \neq n - j + 1$, then $\gamma_{sleat}(G) = 3$.*
2. *If m is odd, $i = j$, and $i \notin \{1, n\}$, then $\gamma_{sleat}(G) = 4$.*

Proof. By Theorem 2.1, we have f, SLEAT labelings of path graph. Suppose g is SLEAT labelings of the connected multiple path.

Case 1. n is even, $i \neq j$, and $i \neq n - j + 1$

Since $\Delta(G) = 3$, $\gamma_{sleat}(G) \geq 3$. Next, we will show that $\gamma_{sleat}(G) \leq 3$. Since the graph has reflexive form, if $i = 1$ we can redefine $(i, j) = (1, j')$ into $(i, j) = (j', 1)$, thus we only consider $i \geq 2$. Add edges $v_n^k v_1^{k+1}$ for every integer $k \in [1, m]$, forming P_{mn}, apply f in a path from v_1^1 to v_n^m. Now, for every integer $k \in [1, m-1]$, g is a labeling that follows

$$g(v_i^k v_j^{k+1}) = f(v_n^k v_1^{k+1}),$$

$$g(V(G) = f(V(P_{mn})),$$

$$g(E(G \backslash (\cup_k \{v_i^k v_j^{k+1}\}) = f(E(P_{mn} \backslash (\cup_k \{v_n^k v_1^{k+1}\}).$$

In the proof of Theorem 2.4, we have shown that $w(v_i^1 v_j^2)$ with two other weights in the paths, with restriction of $i \neq n - j + 1$. Hence, we have the weights

$$w(v_i^k v_{i+1}^k) = \begin{cases} \frac{5}{2}nm - 1, & \text{if } \; i \text{ is even,} \\ \frac{5}{2}nm + 1, & \text{if } \; i \text{ is odd.} \end{cases}$$

Originally, $f(v_n^k) + f(v_n^k v_1^{k+1}) + f(v_1^{k+1})$ is constant for every integer $k \in [1, m-1]$. Since $g(v_n^k) - g(v_i^k)$ and $g(v_1^{k+1}) - g(v_j^{k+1})$ are constant for the same k, we have third distinct weight $w(v_i^k v_j^{k+1})$. Hence, $\gamma_{sleat}(G) = 3$.

Case 2. m is odd, $i = j$, and $i \notin \{1, n\}$

Since $\Delta(G) = 4$, $\gamma_{sleat}(G) \geq 4$. Next, we will show that $\gamma_{sleat}(G) \leq 4$. Add edges $v_n^k v_n^{k+1}$ for every odd integer $k \in [1, m-2]$ and $v_1^k v_1^{k+1}$ for every even integer $k \in [2, m-1]$, forming P_{mn}, apply f in a path from v_1^1 to v_n^m. Now, for every integer $k \in [1, m-1]$ and integer $k' \in [1, \frac{m-1}{2}]$, g is a labeling that follows

$$g(v_i^k v_i^{k+1}) = \begin{cases} f(v_n^k v_n^{k+1}) & \text{for} \quad k \text{ is odd}, \\ f(v_1^k v_1^{k+1}) & \text{for} \quad k \text{ is even}, \end{cases}$$

$$g(V(G)) = f(V(P_{mn})),$$

$$g(E(G \backslash (\cup_k \{v_i^k v_i^{k'+1}\}))) = g(E(P_{mn} \backslash (\cup_k' \{v_1^{2k'} v_1^{2k'+1}, v_n^{2k'-1} v_n^{2k'}\}))).$$

Since $f(v_1^{2k'} v_1^{2k'+1}) - f(v_n^{2k'-1} v_n^{2k'})$ is constant for every $k' \in [1, \frac{m-1}{2}]$, $w(v_1^{2k'} v_1^{2k'+1})$ is constant for every k, $w(v_n^{2k'-1} v_n^{2k'})$ is constant for every k, and $w(v_1^{2k'} v_1^{2k'+1}) \neq w(v_n^{2k'-1} v_n^{2k'})$. Hence, $\gamma_{sleat}(G) = 4$.

Therefore, the theorem holds. $\square$

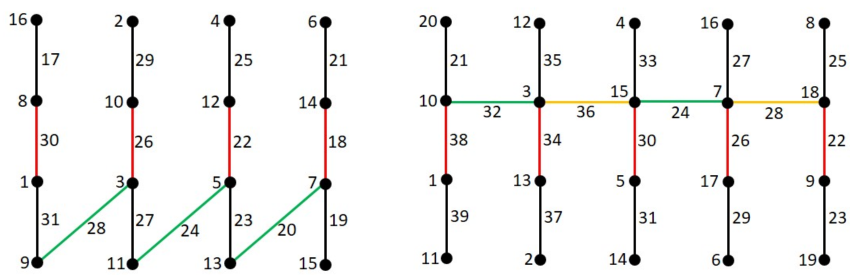

Figure 5: SLEAT coloring of $Hr(4, 4, 4, 3)$ and $Hr(5, 4, 2, 2)$

The same procedure can also be applied if $i \in \{1, n\}$ for G, but it will not give an exact value of $\gamma_{sleat}(G)$ since $\gamma(G)$ is still lower than the number of the weights. Despite the uncertainty, we establish the corollary as follows.

Corollary 2.1. *Let integer $n \geq 2$ and odd integer $m \geq 3$. If $G \cong Hr(m, n, i, i)$, then $3 \geq \gamma_{sleat}(G) \geq 4$*

It can be seen that the graph from the corollary is isomorphic with the comb product of P_m with P_{n-1}. We revised a theorem that given by Kurniawati et al. in [7].

We attempted to find the value for some (m, n, i, j) that did not satisfy preceding theorem premises, but it is not completed yet. It is left as an open problem.

Open Problem 1. *Find the value of $\gamma_{sleat}(Hr(m, n, i, j))$ for remaining cases of (m, n, i, j).*

2.4. Amalgamation of Star

We proceed by finding SLEAT for star graph. Star graph is denoted as S_m where $V(S_m) = \{c, v_i | 1 \leq i \leq m\}$ and $E(S_m) = \{cv_i | 1 \leq i \leq m\}$. Since every edge in the graph is adjacent, $\gamma(S_m) = m$. Since $\gamma(G) \leq \gamma_{sleat}(G) \leq |E(G)|$ for arbitrary graph G, it is clear that $\gamma_{sleat}(S_m) = m$. If one want an example how to achieve such graph, labels as follows $f(v_i) = i, f(c) = m+1, f(cv_i) = m+i+1$. The weights is going to be consecutive by 2 as the common difference.

In this section, we study an amalgamation of star and graph which have certain γ_{sleat}.

Theorem 2.6. *Suppose G_1 be a SLEAT colorable graph and $G_2 \cong Amal(S_m, G_1; c)$ which vertex c is a center in S_m, for integer $m \geq 1$. $\gamma_{sleat}(G_2) \leq \gamma_{sleat}(G_1) + m$*

Proof. Let $n_v = |V(G_1)|$ and $n_e = |E(G_1)|$. Suppose g is the SLEAT labelings of G_1 and f is the SLEAT labelings of G_2. Labels as follows:

$$g(V(G_1)) = f(V(G_1)),$$

$$g(E(G_1)) = f(E(G_1)) + m,$$
$$g(v_i) = n_v + i,$$
$$g(cv_i) = n_v + n_e + m + i.$$

Let w_f is weights as a result of f. Now, we define $w'_f = w_f + m$. Next, we define w_s as $w_s = g(v_i) + g(cv_i) + g(c) = 2n_v + n_e + m + 2i + f(c)$. Now, every distinct w'_f and every distinct w_s are weights that is generated from g. What is left to prove is $w'_f \neq w_s$ for arbitrary possible weights.

Let p is the lowest $f(c)$ for arbitrary c. Since p is adjacent with any w'_f, we can observe the upper bound for w'_f by assuming w'_f contains the largest vertex label n_v, and largest edge label $n_v + n_e + m$. As a result,

$$w'_f \leq n_v + n_v + n_e + m + p = 2n_v + n_e + m + p$$

Now, we can find the lower bound any w_s, by choosing the lowest possible $i = 1$, then

$$w_s \geq 2n_v + n_e + m + p + 2$$

It is clear that $2n_v + n_e + m + p + 2 \geq 2n_v + n_e + m + p$. Therefore, $w'_f \neq w_s$ for arbitrary possible weights. This holds the theorem. □

The preceding theorem applies for any vertex c in graph G_1. By choosing certain c, we have Corollary 2.2.

Corollary 2.2. *Let G_1 is a SLEAT colorable graph and $G_2 \cong Amal(S_m, G_1; c)$ which vertex c is a center in S_m, for integer $m \geq 1$. If $\deg(c) = \Delta(G_2), \gamma_{sleat}(G_2) = \gamma_{sleat}(G_1) + m$.*

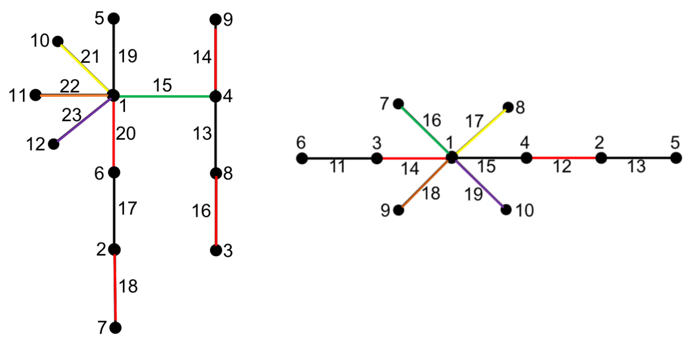

Figure 6: SLEAT coloring of $Amal(S_3, Hd(5,4,2,2), c)$ and $Amal(S_4, P_6, c)$

3. Summary

Here we summarize our results in Table 1.

Table 1: Summary

Graph	Notation	γ_{sleat}	Condition
Path	P_n	2	
Path + edge	$P_n + v_i v_j$	3	$1 \leq i < j \leq n$, and $(i,j) \neq (1,n)$
Path + multiple edges	$P_n + \{v_{i-k}v_{j-k} \| 0 \leq k \leq p\}$	3	$1 \leq i < j \leq n$, $i+j$ is odd, $0 \leq p \leq \min\{i-1+\text{even}(n), j-i-1\}$ and $(i,j) \neq (1,n)$
Hedge graph	$Hd(m,n,i,j)$	3	$v_i \in V(P_n)$, $u_j \in V(P_m)$, and $(i,j) \notin (1,n) \times (1,m)$
Hedgerow graph	$Hr(m,n,i,j)$	3	n is even, $i \neq j$, and $i \neq n-j+1$
Hedgerow graph	$Hr(m,n,i,j)$	4	m is odd, $i = j$, and $i \notin \{1,n\}$
Star	S_n	n	
Amalgamation of star	$Amal(G, S_n; c)$	at most $\gamma_{sleat}(G)+n$	$c \in V(G), V(S_n)$ and c is center in S_n
Amalgamation of star	$Amal(G, S_n; c)$	$\gamma_{sleat}(G)+n$	$c \in V(G), V(S_n)$, c is center in S_n, and $\deg(c) = \Delta(G)$

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